Meeting date : 9/30
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TEXTBOOK CIRCLE: BLACK HOLE ASTROPHYSCS

Ch6

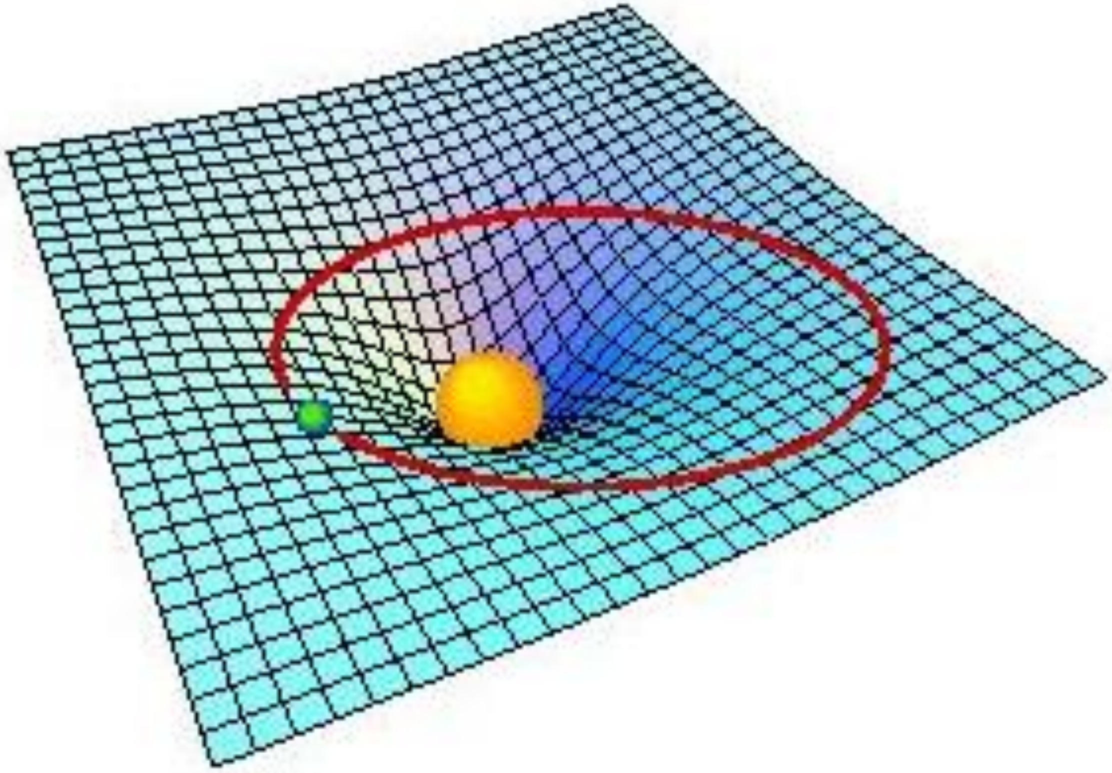
Geometry and Physics without Gravity: Special Relativity

CH6.1 WHY GEOMETRY?

- Since Albert Einstein published his theories of special and general relativity,in 1905 and 1915,it has been recognized that geometry is important not only to mathematics,but also to physics as well.
- Two Major reason for this.

CH6.1 WHY GEOMETRY?

- First, it has become very apparent that the fundamental number of dimensions that describe our(macroscopic) world is four , not three.
- Second, Geometry is important to physics is the discovery that gravity is the result of matter bending the geometry of space and time , so that it is no longer the “flat” geometry of special relativity.



CH6.2 TWO-DIMENSIONAL PYTHAGOREAN GEOMETRY

- Line Element:

The most fundamental equation of plane geometry is the Pythagorean theorem for a right triangle $h^2 = x^2 + y^2$

x and y are two sides of the triangle and h is the hypotenuse.

The distance Δh between two points on a graph

$$(\Delta h)^2 = (\Delta x)^2 + (\Delta y)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

The very small coordinate distance

$$dh^2 = dx^2 + dy^2$$

This equation is called line element

CH6.2.2 LINE ELEMENT HAS THE SAME LENGTH IN ANY COORDINATES

- Example

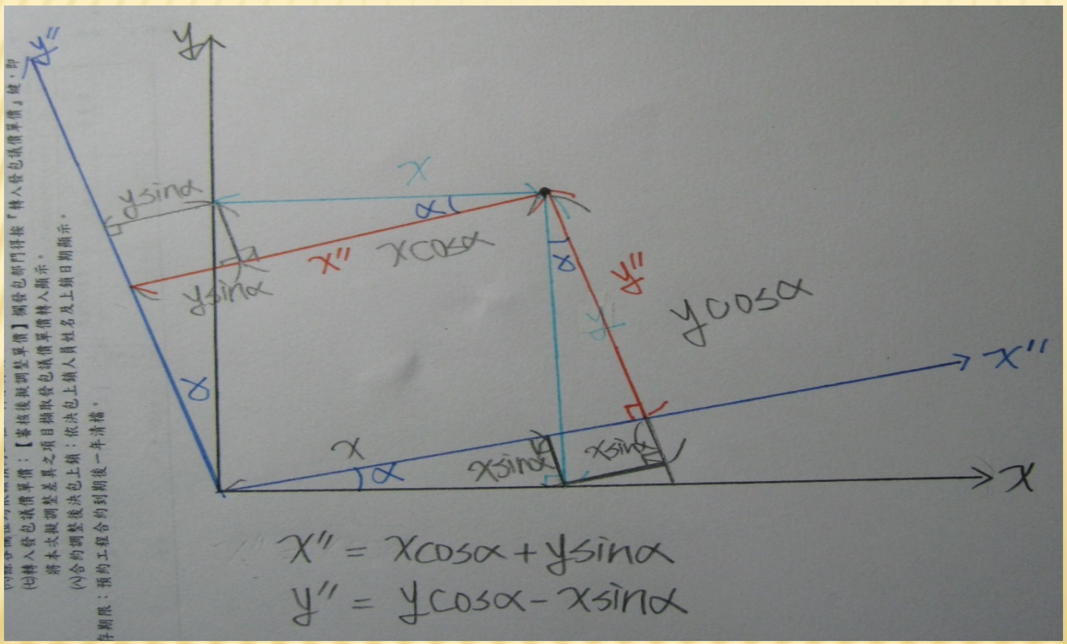
A translation of the origin to a new location (x_0, y_0) produces the coordinate transformation $x' = x - x_0, y' = y - y_0$, the dh still given by

$$(dh')^2 = (dx')^2 + (dy')^2 = (dx)^2 + (dy)^2 = (dh)^2$$

dx_0 and dy_0 are be zero

CH6.2.2 LINE ELEMENT HAS THE SAME LENGTH IN ANY COORDINATES

- A rotation of the axes by a constant angle α



$$
\begin{aligned}
(dh'')^2 &= (dx'')^2 + (dy'')^2 \\
&= (dx)^2 \cos^2\alpha + 2\,dx\,dy \sin\alpha\cos\alpha + (dy)^2 \sin^2\alpha + \\
&\quad (dx)^2 \sin^2\alpha - 2\,dx\,dy \sin\alpha\cos\alpha + (dy)^2 \cos^2\alpha \\
&= dx^2 + dy^2 = dh^2
\end{aligned}
$$

CH6.2.2 LINE ELEMENT HAS THE SAME LENGTH IN ANY COORDINATES

- From Cartesian(x,y) to Polar (R,Ø)

$x = R\cos\emptyset \quad y = R\sin\emptyset$

$$\begin{aligned} dx &= \cos\phi\, dR - R\sin\phi\, d\phi \\ dy &= \sin\phi\, dR + R\cos\phi\, d\phi \end{aligned}$$

Combine equation to $(dh)^2 = (dx)^2 + (dy)^2$

Get follow

$$\begin{aligned} dh^2 = \cos^2\phi\, dR^2 - 2R\sin\phi\cos\phi\, dRd\phi + R^2\sin^2\phi\, d\phi^2 + \\ \sin^2\phi\, dR^2 + 2R\sin\phi\cos\phi\, dRd\phi + R^2\cos^2\phi\, d\phi^2 \\ = dR^2 + R^2\, d\phi^2 \end{aligned}$$

Therefore, Geometry remains unchanged during any coordinate transformation.

MATRM FORM FOR THE GEOMETRY EQUATIONS

- The Metric

It will help our to illustrate the concept of matrics,vectors,1-form,and tensors.

Equation $(dh)^2=(dx)^2+(dy)^2$ can be written

$$(dh)^2=dx^T g\,dx$$

Where $dx=\begin{pmatrix} dx \\ dy \end{pmatrix}$, $dx^T=\begin{pmatrix} dx & dy \end{pmatrix}$,

$g=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is metric of a flat two-dimensional plane in Cartesian coordinates.

MATRM FORM FOR THE GEOMETRY EQUATIONS

Similarly equation $dh^2 = dR^2 + R^2\, d\phi^2$ can be written in matrix form as $dh^2 = dx''^{T}\, g''\, dx''$

Where $dx'' = \begin{pmatrix} dR \\ d\phi \end{pmatrix}$, $g'' = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}$

g'' is the matric of a flat plane but expressed in polar coordinate.

COORDINATE TRANSFORMATIONS: VECTORS AND 1-FORM

For the conversion from Cartesian to polar coordinates, we have

$$dx^{''} = L \cdot dx$$

The coordinate transformation matrix L is

L=$\frac{1}{R}\begin{pmatrix} Rcos\emptyset & Rsin\emptyset \\ -sin\emptyset & cos\emptyset \end{pmatrix}$

The invrse coordinate transformation also can be expressed in matrix form

$$dx = L^{-1}\, dx^{''}$$

Where

$$L^{-1} = \begin{pmatrix} cos\emptyset & -Rsin\emptyset \\ -sin\emptyset & cos\emptyset \end{pmatrix},\; L^{-1} \cdot L = I$$

L is a differential transformation that operates on local vectors and 1-forms

COORDINATE TRANSFORMATIONS: VECTORS AND 1-FORM

In equation $dx^{''} = L \cdot dx$, dx is called a contravariant vector or vector

In the transformation equation for the components of any vector V, the matrix of the vector's $V^{''}$

$$V^{''} = L \cdot V$$

$V^{''}$: the new coordinate system V: the old system

$$V = \begin{pmatrix} V^{x} \\ V^{y} \end{pmatrix}$$

Where V is contravariant vector (superscripts)

COORDINATE TRANSFORMATIONS: VECTORS AND 1-FORM

The contravariant vector whoes components transform in the opposite manner to those of a contravariant vector

$\nu\uparrow'' = \upsilon \cdot L\uparrow -1$, $\nu\uparrow''$ is new coordinate system

υ and V are different geometric quantities that transform in different ways.

DOT PRODUCTS AND MAGNITUDES OF VECTORS

- How to compute the magnitude of all vectors V?

$V^2 = V^T \cdot g \cdot V$ and $v \equiv V^T \cdot g$

Transforms according to equation $\nu'' = v \cdot L^{-1}$

Rewrite $V^2 = v \cdot V$

Use $(AB)^T = B^T A^T$ to $v = V^T \cdot g$, get $V = g^{-1} \cdot v^T$

$$V^2 = v \cdot g^{-1} \; v^T$$

TENSORS

- Discussion of two-dimensional geometry by looking at how the metric itself transforms.

Combine equation $dh^2 = dx^T g\, dx$, $dh^2 = dx'^{\,2}\, g\, dx'$, $dx' = L \cdot dx$ to solve for g'' in term of g to get $g'' = (L^{-1})^T \cdot g \cdot L^{-1}$

g is called covariant tensor of rank 2

$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, it has a special property that is always symmetric $g^T = g$

g is not called a 2-form , because true 2-forms must possess another property- antisymmetry

TENSORS

Matrix can be converted to a contravariant form by taking its inverse

$(g^{''})^{-1} = ((L^{-1})^{T} \cdot g \cdot L^{-1})^{-1}$, use $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

Get $(g^{''})^{-1} = L \cdot g^{-1} \cdot L^{T}$

In general we can get covariant tensor t into contrvariant T vector form is $T = g^{-1} \cdot t \cdot g^{-1}$

g^{-1} is also the corresponding 2-tensor for g by

G= $g^{-1} \cdot g \cdot g^{-1}$ = $g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$

A WORD OF WARNING: GLOBAL VS LOCAL TRANSFORMATIONS

- Global coordinate transformation is one can express the new coordinates as complete and independent functions of the old coordinates.
- Example

$$x' = f(x,y) \quad y' = g(x,y)$$

f and g being independent function of the old coordinates.

A global transform can be written in a differential or matrix manner

$$dx' = a(x,y)dx + b(x,y)dy\,,\; dy' = c(x,y)dx + d(x,y)dy$$

These expressions must be total differentials, expressible in the form

$$dx' = d[f(x,y)]\,, dy' = d[g(x,y)]$$

Global transformations allow the points in an entire space to be re-label

A WORD OF WARNING: GLOBAL VS LOCAL TRANSFORMATIONS

- Local transformation is one that may be expressed in differential or matrix form, but cannot be integrated into the global form.

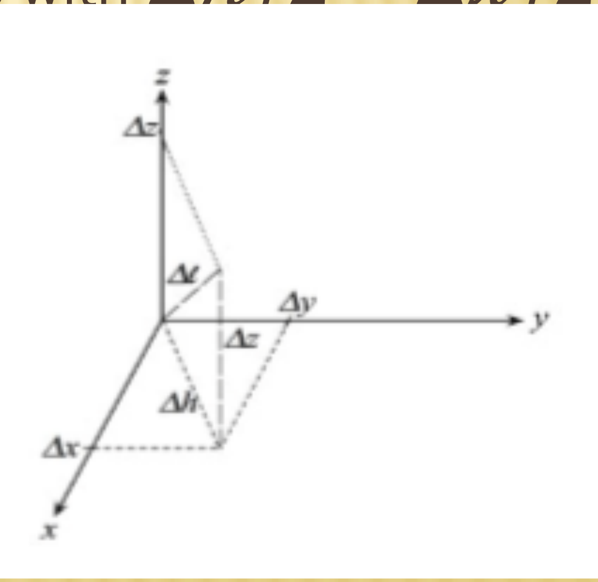
A simple example would be following transformation from polar coordinates (R,Ø) local to a local orthonormal system ($x\uparrow'$,$y\uparrow'$) at a single point $dx\uparrow' = dR$, $dy\uparrow' = Rd\text{Ø}$

THREE-DIMENSIONAL EUCLIDEAN GEOMETRY

- According to previous inference , we can get more general result at following:
- In this plane the Pythagorean theorem(Fig6.4)

$\Delta l^{2} = \Delta h^{2} + \Delta z^{2}$ combine this with $\Delta h^{2} = \Delta x^{2} + \Delta y^{2}$

We get $\Delta l^{2} = \Delta x^{2} + \Delta y^{2} + \Delta z$



THREE-DIMENSIONAL EUCLIDEAN GEOMETRY

- cylindrical coordinate system:

$$dl^2 = dR^2 + R^2\, d\phi^2 + dz^2$$

- spherical-polar coordinate system:

$$dl^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

- The magnitude:

$$V^2 = V^T \cdot g \cdot V = v \cdot g^{-1}\, v^T$$

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thank you for listening